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ON THE SPECTRAL CHARACTERISTICS OF SIGNLESS LAPLACIAN MATRIX

Pallabi Bora and Muktarul Rahman*

Department of Mathematics,
Cotton University,
Guwahati - 781001, Assam, INDIA

E-mail : borapallabi47@gmail.com

*Department of Mathematics,
Gauhati University,
Guwahati - 781014, Assam, INDIA

E-mail : mrahman23.math@gmail.com

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Abstract: In this paper, we present a comprehensive study on the spectral properties of the signless Laplacian matrix of the maximal graph. Specifically, we characterize the spectral radius of the signless Laplacian matrix of the maximal graph $M(\Gamma(\mathbb{Z}_n))$. Moreover, we study the smallest signless Laplacian eigenvalue of the maximal graph and introduce an interaction with the algebraic connectivity of $M(\Gamma(\mathbb{Z}_n))$ for some definite values of n. Finally, we derive an explicit formula for the Wiener index in terms of signless Laplacian eigenvalues of the graph.

Keywords and Phrases: Signless Laplacian Spectrum, Maximal Graph, Wiener Index.

2020 Mathematics Subject Classification: 05C50, 05C25, 05C75.

1. Introduction

In this paper, we consider only undirected simple graph $G(V, E)$, with vertex set V and edge set E and we denote two vertices v_i and v_j are adjacent by $v_i \sim v_j$. Adjacency matrix of a graph G is defined as $A(G) = (a_{ij})_{n\times n}$, $a_{ij} = 1$ and 0 according as v_i is adjacent to v_j or not. The difference between diagonal matrix $D(G)$, with diagonal entries is the degree of the corresponding vertices, and the adjacency matrix $A(G)$ is said to be Laplacian matrix $L(G)$, therefore $L(G) = D(G) - A(G)$ and the signless Laplacian matrix $Q(G)$ is defined as $D(G)+A(G)$. The Laplacian matrix, a crucial matrix representation of a graph, plays a vital role in understanding various properties of the graph. Recently, the signless Laplacian matrix has gained significant attention due to its unique characteristics and applications. Some of them are seen in [7, 8]. The signless Laplacian energy of a graph was introduced by Pirzada et al. [9]. They obtained the upper bounds for the signless Laplacian energy of a graph and characterized the extremal cases. Pirzada et al. [10] determined the signless Laplacian eigenvalues of the zero divisor graph $\Gamma(\mathbb{Z}_n)$ for $n = p^{M_1}q^{M_2}$, where $p < q$ are primes and M_1, M_2 are positive integers. Gaur and Sharma [5] introduced the maximal graph $M(\Gamma(R))$, corresponding to the non-unit elements of R. It is a graph with vertices being the non-unit elements of R, where two distinct vertices a and b are adjacent if and only if there is a maximal ideal of R containing both. In this paper, we emphasize on the significant of spectral characteristics of the signless Laplacian matrix of maximal graph $M(\Gamma(\mathbb{Z}_n))$, of $\mathbb{Z}_n$, the ring of integer modulo n.

The signless Laplacian spectrum of G is the set of eigenvalues (with multiplicity) of the signless Laplacian matrix of G. We denote the distinct signless Laplacian eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ of G with multiplicity $m_1, m_2, \ldots, m_k$ by

$$\sigma_Q(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_k \\ m_1 & m_2 & \ldots & m_k \end{pmatrix}.$$

The signless Laplacian eigenvalues of the complete graph K_n is given by,

$$\sigma_Q(K_n) = \begin{pmatrix} 2n-2 & n-2 \\ 1 & n-1 \end{pmatrix}.$$

The Laplacian spectral radius and the signless Laplacian spectral radius of G are denoted by $\lambda(G)$ and $\lambda_Q(G)$, respectively. The algebraic connectivity of a graph G is defined as the second smallest Laplacian eigenvalue of G and is denoted by $\mu(G)$. Fiedler [4] has proved that $\lambda(G) = n - \mu(\overline{G})$, where $\overline{G}$ is the complement of graph G. The vertex connectivity $\kappa(G)$, is defined as the minimum number of vertices of G, which need to be removed from $V(G)$, so that the induced subgraph of G obtained after removing the vertices is disconnected or has only one vertex. Chattopadhya et al. [3] have introduced some particular values of n for which the algebraic connectivity $\mu(G)$, of the zero divisor graph coincides with the corresponding vertex connectivity $\kappa(G)$. The smallest signless Laplacian eigenvalue of G is denoted by $\mu_Q(G)$. In [11], Schwenk defined the G-generalised join graph or the generalised composition graph, which plays a significant role to find the signless Laplacian spectrum and normalized Laplacian spectrum of $Z^*(\Gamma(\mathbb{Z}_n))$ for different values of n. For a graph G with the vertex set $V(G) = \{v_1, v_2 \dots v_k\}$, the G- generalized join graph $G[H_1, H_2 \dots H_k]$ of k pairwise disjoint graphs $H_1, H_2 \dots H_k$ is the graph formed by replacing each vertex v_i of G by the graph H_i and then joining each vertex of H_j whenever $v_i \sim v_j$ in G.

In this paper, we study the signless Laplacian spectrum of maximal graph $M(\Gamma(\mathbb{Z}_n))$, for some particular values of n and prove that $M(\Gamma(\mathbb{Z}_n))$ is a signless Laplacian integral if and only if $\lambda_Q(M(\Gamma(\mathbb{Z}_n)))$ is an integer. Also, we characterize some values of n, for which the smallest signless Laplacian eigenvalue $\mu_Q(M(\Gamma(\mathbb{Z}_n)))$ coincides with algebraic connectivity and vertex connectivity. In section-2, we prove that $\mu(M(\Gamma(\mathbb{Z}_n))) = n - \lambda_Q(\overline{M(\Gamma(\mathbb{Z}_n))})$ if and only if n is a product of two distinct prime numbers. A distance - based topological index, Wiener index $W(G)$ of G is defined as the sum of the distances of all the distinct pair of vertices of G. In section-3, we introduce the significance of the signless Laplacian spectrum on the Wiener index of maximal graph $M(\Gamma(\mathbb{Z}_n))$.

2. Main Result

A graph G is said to be a signless Laplacian integral if all signless Laplacian eigenvalues are integers. In Theorem 2.4, we recognize the significance of signless Laplacian spectral radius in signless Laplacian integral of $M(\Gamma(\mathbb{Z}_n))$. We shall rapidly use the following Lemma 2.1 and Lemma 2.2 from [1].

Lemma 2.1. *Every elements of A_{d_i} are adjacent to all elements of A_{d_j} in $M(\Gamma(\mathbb{Z}_n))$ if and only if d_i and d_j both are contained in the same maximal ideal of $\mathbb{Z}_n$.*

From [13], we have, $|A_{d_i}| = \phi\left(\frac{n}{d_i}\right)$.

Lemma 2.2. *For every positive integer n,*

$$M\left(\Gamma(\mathbb{Z}_n)\right) = \delta_n\left[K_{\phi(\frac{n}{d_1})}, K_{\phi(\frac{n}{d_2})}, \dots, K_{\phi(\frac{n}{d_k})}\right],$$

where δ_n is a graph with vertices being the proper divisors of n, two vertices x and y are adjacent in δ_n if and only if x and y both are contained in the same maximal ideal of $\mathbb{Z}_n$.

The following Theorem 2.3 is proved in [12].

Theorem 2.3. *Let G be a graph with vertex set $\{v_1, v_2, \dots, v_k\}$ and let $H_1, \dots H_k$ be $k-$pairwise disjoint r_1- regular,..., r_k- regular graphs with $m_1, \dots m_k$ vertices, respectively. Then the signless Laplacian spectrum of $G\left[H_1, H_2, \dots H_k\right]$ is given by*

$$\sigma_Q\left(G[H_1, H_2, \ldots H_k]\right) = \left(\bigcup_{j=1}^{k}(M_j + (\sigma_Q(H_j) \setminus \{2r_j\}))\right) \bigcup \sigma(\mathcal{Q}(G)), \qquad (1)$$

where $\mathcal{Q}(G) = (q_{ij})_{k\times k}$ *with*

$$q_{ij} = \begin{cases} 2r_i + M_i, & i = j \\ \sqrt{m_i m_j}, & v_i \sim v_j \\ 0, & otherwise \end{cases}, \qquad (2)$$

and

$$M_j = \begin{cases} \sum_{v_i \sim v_j} m_i, & \text{if } N_G(v_j) \neq \phi, \\ 0, & otherwise \end{cases}.$$

In Equation (1), $\sigma_Q(H_j) \setminus \{2r_j\}$ *means that one copy of eigenvalue* $2r_j$ *is removed from the multi-set* $\sigma_Q(H_j)$ *and* $M_j + \sigma_Q(H_j) \setminus \{2r_j\}$ *means that* M_j *is added to each element of* $\sigma_Q(H_j) \setminus \{2r_j\}$*.*

Theorem 2.4. *For* $n = pq$, p^α, $p^\alpha q^\beta$ *where* p *and* q *are distinct prime numbers and* α *and* β *are two positive integers with* $\alpha < \beta$*, the maximal graph* $M(\Gamma(\mathbb{Z}_n))$ *is a signless Laplacian integral if and only if its spectral radius,* $\lambda_Q(M(\Gamma(\mathbb{Z}_n)))$ *is an integer.*

Proof. By using Lemma 2.2 $M(\Gamma(\mathbb{Z}_{pq})) = K_{\phi(q)} \cup K_{\phi(p)}$. Therefore, the signless Laplacian characteristic polynomial of $M(\Gamma(\mathbb{Z}_{pq}))$ is given by

$$(x - (2\phi(q) - 2))(x - (2\phi(p) - 2))(x - (\phi(q) - 2))^{\phi(q)-1}(x - (\phi(p) - 2))^{\phi(p)-1}.$$

Thus

$$\sigma_Q(G) = \begin{pmatrix} 2\phi(q) - 2 & 2\phi(p) - 2 & \phi(q) - 2 & \phi(p) - 2 \\ 1 & 1 & \phi(q) - 1 & \phi(p) - 1 \end{pmatrix}.$$

Again for $n = p^\alpha$, $M(\Gamma(\mathbb{Z}_{p^\alpha})) = K_{p^{\alpha-1}-1}$. Hence, the result is obvious for $n = pq$ and p^α.

We assume $n = p^\alpha q^\beta$. Followed from the proof of Lemma 2.7, we have

$$M(\Gamma(\mathbb{Z}_{p^\alpha q^\beta})) = K_m \vee [K_{m_1} \cup K_{m_2}],$$

with $m = \sum_{i=1}^{\alpha}\sum_{j=1}^{\beta}\phi(p^{\alpha-i}q^{\beta-j}) - 1$, $m_1 = \sum_{i=1}^{\alpha}\phi(p^{\alpha-i}q^{\beta})$, $m_2 = \sum_{j=1}^{\beta}\phi(p^{\alpha}q^{\beta-j})$ and $m < m_1 < m_2$. By direct calculation, using Theorem 2.3 from [12] the signless Laplacian characteristic polynomial of $M(\Gamma(\mathbb{Z}_{p^\alpha q^\beta}))$ is given by

$$(x-(m+m_1-2))^{m_1-1}(x-(m+m_2-2))^{m_2-1}(x-(m+m_1+m_2-2))^m(x^2-bx+c), \qquad (3)$$

where $b = 3m+2m_1+2m_2-4$ and $c = 2m^2+2mm_1+2mm_2+4m_1m_2-6m-4m_1-4m_2+4$. From equation (1), it is clear that $M(\Gamma(\mathbb{Z}_{p^\alpha q^\beta}))$ is a signless Laplacian integral if and only if the roots of x^2-bx+c are integers.

Note that if λ_1 and λ_2 are two roots of x^2-bx+c with $\lambda_1 \geq \lambda_2$, $\lambda_1 = \frac{b+\sqrt{b^2-4c}}{2}$ and $m+m_1+m_2-2 < b$. Therefore, $\lambda_Q(M(\Gamma(\mathbb{Z}_{p^\alpha q^\beta})))$ is the largest root of x^2-bx+c.

Suppose that $\lambda_Q(M(\Gamma(\mathbb{Z}_{p^\alpha q^\beta})))$ is an integer. Thus, all the roots of x^2-bx+c are integers. Hence, we get the required result.

By using following Lemma 2.5, we characterize the value of n for which vertex connectivity is equal to the smallest signless Laplacian eigenvalue of $M(\Gamma(\mathbb{Z}_n))$ in Theorem 2.6.

Lemma 2.5. *In $M(\Gamma(\mathbb{Z}_{p^\alpha q^\beta}))$, $\mu_Q(M(\Gamma(\mathbb{Z}_{p^\alpha q^\beta}))) = m+m_1-2$.*

Proof. From Equation (3), we have

$$\sigma_Q(M(\Gamma(\mathbb{Z}_{p^\alpha q^\beta}))) = \begin{pmatrix} m+m_1-2 & m+m_2-2 & m+m_1+m_2-2 & \lambda_1 & \lambda_2 \\ m_1-1 & m_2-1 & m & 1 & 1 \end{pmatrix}.$$

It is clear that $m+m_1-2 < m+m_2-2 < m+m_1+m_2-2$. Claim that $m+m_1-2 < \lambda_2 < \lambda_1$. Here $\lambda_1 = \frac{b-\sqrt{b^2-4c}}{2}$.

Now

$$\begin{aligned}(b-\sqrt{b^2-4c}) - 2(m+m_1-2) &= (3m+2m_1+2m_2-4) - \sqrt{\chi} - 2(m+m_1-2) \\ &= m+2m_2-\sqrt{\chi},\end{aligned}$$

where $\chi = m^2+4m_1^2+4m_2^2+4mm_1+4mm_2-8m_1m_2$.

Again,

$$\begin{aligned}(m+2m_2)^2-\chi &= 8m_1m_2-4m_1^2-4mm_1 \\ &= 4p^{\alpha-1}q^{\beta-1}(p-1)[2q-p-2]+4(p-1) > 0,\end{aligned}$$

Therefore, $\sqrt{\chi} < m+2m_2$ and so $(b-\sqrt{b^2-4c}) > 2(m+m_1-2)$. Hence, $\mu_Q(M(\Gamma(\mathbb{Z}_{p^\alpha q^\beta}))) = m+m_1-2$.

Theorem 2.6. *If $n = 4q$, q is a prime number, then $\kappa(M(\Gamma(\mathbb{Z}_n))) = \mu_Q(M(\Gamma(\mathbb{Z}_n)))$.*

Proof. Here $M(\Gamma(\mathbb{Z}_n)) = K_1 \vee [K_2 \cup K_{2(q-1)}]$, then $\kappa(M(\Gamma(\mathbb{Z}_n))) = 1$. From Equation (1),

$$\sigma_Q(M(\Gamma(\mathbb{Z}_n))) = \begin{pmatrix} 1 & 2q-3 & 2q-1 & \lambda_1 & \lambda_2 \\ 1 & 2q-3 & 1 & 1 & 1 \end{pmatrix}.$$

It is clear that $\lambda_1 > \lambda_2 > 1$. Thus, $\mu_Q(M(\Gamma(\mathbb{Z}_n))) = 1$ and so $\kappa(M(\Gamma(\mathbb{Z}_n))) = \mu_Q(M(\Gamma(\mathbb{Z}_n)))$.

Suppose $n \neq 4q$. Therefore, $M(\Gamma(\mathbb{Z}_n)) = K_m \vee [K_{m_1} \cup K_{m_2}]$ with $m_1 > 2$ and so $m + m_1 - 2 > m$. It is clear for $m < m_1 < m_2$, $\kappa(M(\Gamma(\mathbb{Z}_n))) = m$. Hence, from Lemma 2.5, $\kappa(M(\Gamma(\mathbb{Z}_n))) < \mu_Q(M(\Gamma(\mathbb{Z}_n)))$.

The following Lemma 2.7 was introduced in [1], which determines the Laplacian spectrum of $M(\Gamma(\mathbb{Z}_n))$ for some particular values of n.

Lemma 2.7. *Let $n = p^\alpha q^\beta$ where p and q are distinct prime numbers, and α and β are two positive integers with $\alpha < \beta$. The Laplacian spectrum of $M(\Gamma(\mathbb{Z}_n))$ is given by*

$$\sigma_L(M(\Gamma(\mathbb{Z}_n))) = \begin{pmatrix} 0 & m & m+m_1 & m+m_2 & m+m_1+m_2 \\ 1 & 1 & m_1-1 & m_2-1 & m \end{pmatrix}.$$

In the next Lemma 2.8, we determine the value of n for which the smallest signless Laplacian eigenvalue of $M(\Gamma(\mathbb{Z}_n))$ and its algebraic connectivity are equal.

Lemma 2.8. *If $n = 4q$, q is a prime number, then $\mu(M(\Gamma(\mathbb{Z}_n))) = \mu_Q(M(\Gamma(\mathbb{Z}_n)))$.*
Proof. Here $m = 1, m_1 = 2$ and $m_2 = 2(q-1)$. By using Lemma 2.7, we have

$$\sigma_L(M(\Gamma(\mathbb{Z}_{4q}))) = \begin{pmatrix} 0 & 1 & 3 & 2q-1 & 2q+1 \\ 1 & 1 & 1 & 2q-3 & 1 \end{pmatrix}.$$

Therefore the algebraic connectivity of $M(\Gamma(\mathbb{Z}_{4q}))$, $\mu(M(\Gamma(\mathbb{Z}_{4q}))) = 1$. Thus, the result follows from Lemma 2.5.

The following result, Theorem 2.9, characterizes the values of n for which the complement graph of $M(\Gamma(\mathbb{Z}_n))$ is complete bipartite and join of two graphs. By using Theorem 2.9, we provide an explicit formula to find the algebraic connectivity of $M(\Gamma(\mathbb{Z}_n))$ for some particular values of n in Theorem 2.11.

Theorem 2.9. *$\overline{M(\Gamma(\mathbb{Z}_n))}$ is complete bipartite graph K_{m_1,m_2} if and only if n is a product of two distinct primes with $m_1 = \phi(q)$ and $m_2 = \phi(p)$.*
Proof. For $n = pq$, $M(\Gamma(\mathbb{Z}_{pq})) = K_{\phi(q)} \cup K_{\phi(p)}$. Considering the partition U and V are equal to $V(K_{\phi(q)})$ and $V(K_{\phi(p)})$ respectively. Therefore, from Lemma 2.1, it follows that $\overline{M(\Gamma(\mathbb{Z}_n))}$ is a complete bipartite if n is a product of two distinct primes.

Conversely, suppose that $n = p_1^{r_1} p_2^{r_2} \dots p_k^{r_k}$, where $p_1, p_2, \dots, p_k$ are distinct primes and $r_1, r_2, \dots, r_k, k$ are positive integers. Assume that $k = 2$ with at least one of r_1 and r_2 is greater than 1. Therefore, by Lemma 2.1, p_1p_2 is adjacent to all other vertices in $M(\Gamma(\mathbb{Z}_n))$ and so p_1p_2 is an isolated vertex in $\overline{M(\Gamma(\mathbb{Z}_n))}$. Hence $\overline{M(\Gamma(\mathbb{Z}_n))}$ is disconnected.

For $k \geq 3$, follows from Lemma 2.1, $p_1, p_2, \ldots, p_k$ are mutually adjacent in $\overline{M(\Gamma(\mathbb{Z}_n))}$. Thus, it contains at least one odd cycle. This implies that $\overline{M(\Gamma(\mathbb{Z}_n))}$ cannot be bipartite.

For $k = 1$, $\overline{M(\Gamma(\mathbb{Z}_n))} = \overline{K_{p^{r_1-1}-1}}$. Thus, it is totally disconnected. Hence $\overline{M(\Gamma(\mathbb{Z}_n))}$ is complete bipartite graph if and only if n is a product of two distinct primes.

Lemma 2.10. (Fiedler [4]) *If G is a graph on n vertices, then $\lambda(G) = n - \mu(\overline{G})$ if and only if $\overline{G}$ is disconnected if and only if G is the join of two graphs.*

From the above Theorem 2.9, it directly implies that $\mu(G) = n - \lambda(\overline{G})$ if and only if G is disconnected if and only if $\overline{G}$ is the join of two graphs.

Theorem 2.11. $\mu(M(\Gamma(\mathbb{Z}_n))) = n - \lambda_Q(\overline{M(\Gamma(\mathbb{Z}_n))})$ *if and only if n is a product of two distinct primes.*

Proof. Follows from Theorem 2.9 $\overline{M(\Gamma(\mathbb{Z}_n))}$ is the join of two graphs if and only if n is a product of two distinct primes. We have $\overline{M(\Gamma(\mathbb{Z}_{pq}))} = K_{\phi(p),\phi(q)} = \overline{K_{\phi(p)}} \vee \overline{K_{\phi(q)}}$. Therefore, by using Lemma 2.10, $\mu(M(\Gamma(\mathbb{Z}_n))) = n - \lambda(\overline{M(\Gamma(\mathbb{Z}_n))})$ if and only if n is a product of two distinct primes. Note that the characteristic polynomial of $L(G)$ and $Q(G)$ are same if and only if G is bipartite. Thus, for $n = pq$, $\lambda_Q(\overline{M(\Gamma(\mathbb{Z}_n))}) = \lambda(\overline{M(\Gamma(\mathbb{Z}_n))})$. Hence, $\mu(M(\Gamma(\mathbb{Z}_n))) = n - \lambda_Q(\overline{M(\Gamma(\mathbb{Z}_n))})$ if and only if n is a product of two distinct primes.

3. Signless Laplacian spectrum and Wiener index of the maximal graph

In [2], Bora and Rajkhowa explained the significances of Laplacian spectrum on the Wiener index of extension of zero divisor graph. In this section, we introduce an interaction of the Wiener index with the signless Laplacian eigenvalues of $M(\Gamma(\mathbb{Z}_n))$ for some particular values of n. We state Lemma 3.1 and Lemma 3.2 from [1].

Lemma 3.1. *For any two vertices x and y, in $M(\Gamma(\mathbb{Z}_n))$, $d(x, y)$ is given by*

$$d(x,y) = \begin{cases} 1; & \text{if } x \text{ is adjacent to } y, \\ 2; & \text{if } x \text{ is not adjacent to } y \end{cases}.$$

From Lemma 2.2, observe that for $n = pq$, $d(x, y) = \infty$ if x is not adjacent to y. As a result, the graph is disconnected.

Lemma 3.2. *The following holds for the diameter of $M(\Gamma(\mathbb{Z}_n))$:*

(i) $diam(M(\Gamma(\mathbb{Z}_n))) = 1$ if and only if $n = p^t$.

(ii) $diam(M(\Gamma(\mathbb{Z}_n))) = 2$ if and only if $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$, with $k \geq 2$ and p_i's are distinct primes and α_i's are positive integers for $i = 1, 2, \ldots, k$.

Theorem 3.3. *The Wiener index of* $M(\Gamma(\mathbb{Z}_n)) = \frac{1}{2}\sum_{i=1}^{m}\lambda_i$, λ_i *are the signless Laplacian eigenvalues of* $M(\Gamma(\mathbb{Z}_n))$ *if and only if* n *is a product of two distinct primes or* n *is a prime power.*

Proof. We know that

$$\begin{aligned} W(G) &= \frac{1}{2}\sum_{u\in V(G)} d(u|G) \\ &= \frac{1}{2}\sum_{u\in V(G)}\left[\sum_{v\in V(G)} d(u,v)\right] \\ &= \frac{1}{2}\sum_{u\in V(G)}\left[\sum_{u\sim v} d(u,v) + \sum_{u\nsim v} d(u,v)\right] \\ &= \frac{1}{2}\left[\sum_{u\in V(G)}(\sum_{u\sim v} d(u,v)) + \sum_{u\in V(G)}(\sum_{u\nsim v} d(u,v))\right]. \end{aligned}$$

Now, $\sum_{u\sim v} d(u,v)) = deg(u) = q_{uu}$, and so $\sum_{u\in V(G)}(\sum_{u\sim v} d(u,v)) = tr(Q(G)) = \sum_{i=1}^{m}\lambda_i$, m is the order of $Q(G)$. Thus,

$$W(G) = \frac{1}{2}\left[\sum_{i=1}^{m}\lambda_i + \sum_{u\in V(G)}(\sum_{u\nsim v} d(u,v))\right]. \tag{4}$$

By Lemma 3.2, $diam(M(\Gamma(\mathbb{Z}_n))) = 1$ if and only if n is a prime power. Thus, every vertices are mutually adjacent in $M(\Gamma(\mathbb{Z}_{p^t}))$. Hence, by Equation (4), $W(M(\Gamma(\mathbb{Z}_{p^t}))) = \frac{1}{2}\left[\sum_{i=1}^{m}\lambda_i\right]$.

Suppose that $n \neq pq$ and $n \neq p^t$, p and q are distinct primes. Therefore, by Lemma 3.2, $diam(M(\Gamma(\mathbb{Z}_n))) = 2$. Thus, there exist v_i and v_j such that $d(v_i, v_j) = 2$ and $q_{v_i v_j} = 0$ in $Q(M(\Gamma(\mathbb{Z}_n)))$. Therefore, by Equation (4), we get

$$W(M(\Gamma(\mathbb{Z}_n))) = \frac{1}{2}\left[\sum_{i=1}^{m}\lambda_i + \sum_{u\in V(M(\Gamma(\mathbb{Z}_n)))}(\sum_{u\nsim v} 2)\right].$$

In [6], Knor et al. have defined the Wiener index for disconnected graph G as the total of Wiener index of its connected components. Here $M(\Gamma(\mathbb{Z}_{pq})) = K_{\phi(p)} \cup K_{\phi(q)}$ and it imply that $W(M(\Gamma(\mathbb{Z}_{pq}))) = W(K_{\phi(p)}) + W(K_{\phi(q)}) = \sum_{\lambda_i\in\sigma_Q(K_{\phi(p)})}\lambda_i + \sum_{\lambda_i\in\sigma_Q(K_{\phi(q)})}\lambda_i$. It is clear that $\sigma_Q(M(\Gamma(\mathbb{Z}_{pq}))) = \sigma_Q(K_{\phi(p)}) \cup \sigma_Q(K_{\phi(q)})$. Hence, the result.

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